ON THE THEORY OF STEADY PROGRESSIVE WAVES ON THE SURFACE OF A FLUID OF INFINITE DEPTH

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In the theory of steady progressive waves of finite amplitude on the surface of a fluid of infinite depth various methods of applying the boundary condition on the wave profile are known. One of these, proposed by Davis [1], consists in transforming this condition to the imaginary part of a differential equation. A defect in the method of Davis is that the condition, expressed in the form of Levi-Civita, is not applied exactly, but is replaced by a condition close to it. In the present paper a method is given for reducing the exact boundary condition to a complex differential equation which permits waves up to forms close to the limiting form to be investigated.

1. We shall consider the motion of a fluid in a system of coordinates $x0y$, stationary relative to the wave profile (Fig. 1). We have the following boundary conditions for the velocity potential and the stream function:

$$\varphi = c\lambda, \quad 0 \leqslant \psi \leqslant \infty \quad \text{on } CD, \qquad \varphi = 0, \quad 0 \leqslant \psi \leqslant \infty \quad \text{on } AE$$
$$0 \leqslant \varphi \leqslant c\lambda, \quad \psi = \infty \quad \text{on } DE, \qquad 0 \leqslant \varphi \leqslant c\lambda, \quad \psi = 0 \quad \text{on } AC \quad (1.1)$$

We shall map the region under consideration in the plane of the complex velocity potential w into a circle of unit radius in the auxiliary

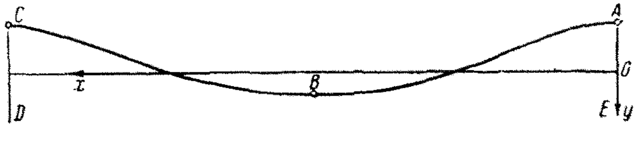

Fig. 1.

plane u with the help of the function

$$w = \frac{\lambda c}{2\pi i} \ln u \tag{1.2}$$

The boundary conditions for the complex conjugate of the velocity will be:

$$\operatorname{Im} \bar{v} = 0 \quad \text{on } CD, \qquad \operatorname{Im} \bar{v} = 0 \quad \text{on } AE, \quad \bar{v} = c \quad \text{on } DE \tag{1.3}$$

For the modulus q of the velocity on AC we have the Bernoulli equation

$$\frac{1}{2} q^2 - gy = C \tag{1.4}$$

where C is a constant. Differentiating with respect to the arc length s of the profile and noting that

$$ds = \frac{d\varphi}{q}; \qquad \frac{dy}{ds} = \sin\theta$$

we obtain

$$q^2 \frac{dq}{d\varphi} - g \sin\theta = 0 \tag{1.5}$$

where θ is the angle of inclination of the tangent of the wave profile to the $0x$ axis.

2. We shall introduce the function

$$\zeta = \tau + i\theta = \ln \frac{c}{\bar{v}} \tag{2.1}$$

where c is the dispersion velocity of the waves. Taking (1.3) and (1.5) into consideration, we obtain the following boundary conditions for the function ζ:

$$\operatorname{Im} \zeta = 0 \quad \text{on } CD, \quad \operatorname{Im} \zeta = 0 \quad \text{on } AE, \quad \zeta = 0 \quad \text{on } DE \tag{2.2}$$

$$\frac{c^3}{g} \frac{\partial \tau}{\partial \varphi} + e^{3\tau} \sin\theta = 0 \quad \text{on } AC \tag{2.3}$$

We shall expand the functions $e^{3\tau}$ and $\sin\theta$ into power series and multiply them. Discarding terms higher than the fifth power and grouping, we obtain

$$e^{3\tau} \sin\theta = \theta + \tfrac{3}{2}(2\tau\theta) + \tfrac{1}{6}(3\tau^2\theta - \theta^3) + \tfrac{1}{8}(4\tau^3\theta - 4\tau\theta^3) +$$
$$+ \tfrac{1}{120}(5\tau^4\theta - 10\tau^2\theta^3 + \theta^5) + 4\tau^2\theta + 4\tau^3\theta + \tfrac{10}{3}\tau^4\theta - \tfrac{2}{3}\tau^2\theta^3 \tag{2.4}$$

We shall write out the imaginary parts of the five powers of ζ

$$\text{Im}\,\zeta = \theta, \qquad \text{Im}\,\zeta^2 = 2\tau\theta, \qquad \text{Im}\,\zeta^3 = 3\tau^2\theta - \theta^3$$
$$\text{Im}\,\zeta^4 = 4\tau^3\theta - 4\tau\theta^3, \qquad \text{Im}\,\zeta^5 = 5\tau^4\theta - 10\tau^2\theta^3 + \theta^5 \tag{2.5}$$

The first five terms on the right-hand side of (2.4) can be represented as

$$\text{Im}\left(\zeta + \tfrac{3}{2}\zeta^2 + \tfrac{1}{6}\zeta^3 + \tfrac{1}{8}\zeta^4 + \tfrac{1}{120}\zeta^5\right) \tag{2.6}$$

It is impossible to represent the rest of the terms as integral powers of ζ. However, as will be shown below, they can be represented as the imaginary part of the power series $F(\chi)$ in the function

$$\chi = \xi + i\eta = \frac{au}{1 - \mu au} \tag{2.7}$$

which is a solution of the differential equation

$$-u\frac{d\chi}{du} + \chi + \mu\chi^2 = 0 \tag{2.8}$$

where a and μ are some real positive constants.

According to (2.4) Im $F(\chi)$ must be equal to zero on DE. Without affecting the generality, we set $F(\chi) = 0$ on DE. Then, writing $\partial\tau/\partial\varphi$ in the form Im $i\,d\zeta/dw$, the condition (2.3) can be represented in the form

$$\text{Im}\left[i\frac{c^3}{g}\frac{d\zeta}{dw} + \zeta + \tfrac{3}{2}\zeta^2 + \tfrac{1}{6}\zeta^3 + \tfrac{1}{8}\zeta^4 + \tfrac{1}{120}\zeta^5 + F(\chi)\right] = 0 \tag{2.9}$$

3. For $a\mu < 1$ the function (2.7) maps a circle of unit radius in the u-plane into a circle of radius R in the χ-plane, which is displaced relative to the initial coordinates by an amount e (Fig. 2). According to (1.2) and (2.8) we have

$$dw = \frac{\lambda c}{2\pi i}\frac{d\chi}{\chi + \mu\chi^2} \tag{3.1}$$

Substituting (3.1) into (2.9), we obtain the condition on the contour of the circle in the χ-plane

$$\text{Im}\left[-\frac{2\pi c^3}{\lambda g}(\chi + \mu\chi^2)\frac{d\zeta}{d\chi} + \zeta + \tfrac{3}{2}\zeta^2 + \tfrac{1}{6}\zeta^3 + \tfrac{1}{8}\zeta^4 + \tfrac{1}{120}\zeta^5 + F(\chi)\right] = 0 \tag{3.2}$$

From the condition of the periodicity of the fluid motion along the $0x$ axis the function ζ takes identical values on both sides of the cut, which is taken on the segment of the real axis lying within the circle of the χ-plane.

The function, appearing in the square brackets in condition (3.2), will also be holomorphic inside the circle of the χ-plane and has an imaginary part which is equal to zero on the contour of the circle.

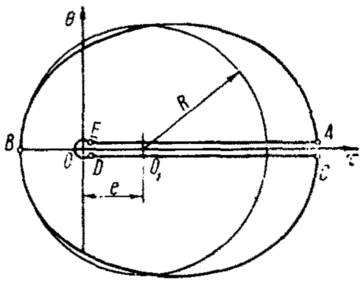

Fig. 2.

Therefore it can be analytically continued into the external part of the circle and it will assume complex conjugate values at points symmetric to the interior points of the circle relative to the contour of the circle. Because this function is holomorphic inside the circle, after the analytic continuation it will then be holomorphic in the entire χ-plane. From the Liouville theorem such a function is equal to a constant. This constant must be set equal to zero if it is taken into consideration that $F(\chi) = 0$, $\zeta = 0$ and $d\zeta/d\chi = 0$ for $\chi = 0$. Thus, in the χ-plane we obtain the equation

$$-\frac{2\pi c^2}{\lambda g}(\chi + \mu\chi^2)\frac{d\zeta}{d\chi} + \zeta + \frac{3}{2}\zeta^2 + \frac{1}{6}\zeta^3 + \frac{1}{8}\zeta^4 + \frac{1}{120}\zeta^5 + F(\chi) = 0 \quad (3.3)$$

The differential equation (3.3) can be integrated, representing the function ζ in the form of the series

$$\zeta = \chi + d_3\chi^3 + d_4\chi^4 + d_5\chi^5 + \ldots \quad (3.4)$$

which according to (2.7) satisfies the boundary conditions (2.2). We obtain the equality $d_2 = 0$ corresponding to the choice of the coefficient μ.

We shall separate the real and imaginary parts in (3.4)

$$\tau = \xi + d_3(\xi^3 - 3\xi\eta^2) + \ldots, \quad \theta = \eta + d_3(3\xi^2\eta - \eta^3) + \ldots \quad (3.5)$$

Hence, discarding terms higher than the fifth power, we find

$$\tau^2\theta = \xi^2\eta + 5d_3\xi^4\eta - 7d_3\xi^2\eta^3$$

$$\tau^3\theta = \xi^3\eta, \quad \tau^4\theta = \xi^4\eta, \quad \tau^2\theta^3 = \xi^2\eta^3 \quad (3.6)$$

$$4\tau^2\theta + 4\tau^3\theta + \frac{10}{3}\tau^4\theta - \frac{2}{3}\tau^2\theta^3 =$$

$$= 4\xi^2\eta + 4\xi^3\eta + \left(\frac{10}{3} + 20d_3\right)\xi^4\eta - \left(\frac{2}{3} + 28d_3\right)\xi^2\eta^3 \quad (3.7)$$

4. In order to represent the right-hand side of the equality (3.7) as the imaginary part of a power series in χ, we determine the values of R and e from Formula (2.7) and write the condition which relates the

functions ξ and η on the contour of the circle

$$(\xi - e)^2 + \eta^2 = R^2 \tag{4.1}$$

Introducing the notation $p = R^2 - e^2$, we rewrite (4.1) in the form

$$\xi^2 = p + 2e\xi - \eta^2 \tag{4.2}$$

Multiplying the equality (4.2) by η; we obtain

$$\xi^2\eta = p\eta + 2e\xi\eta - \eta^3$$

Then, using (2.5), we obtain

$$4\xi^2\eta = \operatorname{Im}(p\chi + e\chi^2 + \chi^3) \tag{4.3}$$

Multiplying the equality (4.2) by $\xi\eta$; we obtain

$$\xi^3\eta = p\xi\eta + 2e\xi^2\eta - \xi\eta^3$$

Then, using (2.5) and (4.3), we find

$$4\xi^3\eta = \operatorname{Im}(ep\,\chi + R^2\chi^2 + e\chi^3 + \tfrac{1}{2}\chi^4) \tag{4.4}$$

Multiplying the equality (4.2) by $\xi^2\eta$ and η^3 in turn

$$\xi^4\eta = p\xi^2\eta + 2e\xi^3\eta - \xi^2\eta^3, \quad \xi^2\eta^3 = p\xi^3 + 2e\xi\eta^3 - \eta^5$$

and taking (2.5), (4.3) and (4.4) into consideration, we obtain (4.5)

$$16\xi^4\eta = \operatorname{Im}[p(2p + 5e^2)\chi + e(2p + 5R^2)\chi^2 + (3p + 5e^2)\chi^3 + 3e\chi^4 + \chi^5$$

$$16\xi^2\eta^3 = \operatorname{Im}[p(2p + 3e^2)\chi + e(2p + 3R^2)\chi^2 + (p + 3e^2)\chi^3 + e\chi^4 - \chi_5$$

It is easily seen that products of $\xi^n\eta^m$ of any power can be represented on the contour of the circle by similar means.

Substituting (4.3), (4.4) and (4.5) into (3.7), we determine $F(\chi)$. Substituting (3.4) into (3.3) and equating coefficients with the same powers of χ, we find after simple transformations

$$c^2 = \frac{\lambda g}{2\pi}(1 + K), \qquad \mu = \frac{\frac{3}{2} + L}{1 + K}, \qquad d_3 = \frac{\frac{7}{6} + e + \frac{1}{12}(7p + 11e^2)}{2 - 2p - e^2 + 3K}$$

$$d_4 = \frac{\frac{5}{8} + \frac{7}{12}e - d_3\left(\frac{3}{2} - 2e + 3L\right)}{3 + 4K}, \qquad d_5 = \frac{\frac{31}{120} + \frac{7}{2}d_3 - d_4(3 + 4L)}{4 + 5K} \tag{4.6}$$

where

$$K = p\,[1 + e + \tfrac{1}{12}(4p + 11e^2) - d_3(p - e^2)]$$

$$L = R^2 + e\,[1 + \tfrac{1}{12}(4p + 11R^2) + d_3 e^2] \tag{4.7}$$

$$\left(R = \frac{a}{1-a^2\mu^2},\quad e = \frac{a^2\mu}{1-a^2\mu^2},\quad p = \frac{a^2}{1-a^2\mu^2}\right)$$

We shall solve the second and third equations of (4.6) for the unknowns μ and d_3 by the method of successive approximations. First, we set $\mu = 3/2$ and compute R, e and p from (4.7). From them, setting $d_3 = 0$ and using formula (4.6), we compute new values of μ and d_3 and repeat the process. From the finally computed values of μ and d_3 we compute all the rest of the quantities.

To construct the domain of the function ζ, in (3.4) we set

$$\chi = e + Re^{i\alpha}$$

As a result we obtain

$$\zeta = a_0 + a_1 e^{i\alpha} + a_2 e^{2i\alpha} + a_3 e^{3i\alpha} + a_4 e^{4i\alpha} \tag{4.8}$$

Here

$$a_0 = e\,[1 + e^2(d_3 + d_4 e + d_5 e^2)],\quad a_1 = R\,[1 + e^2(3d_3 + 4d_4 e + 5d_5 e^2)]$$

$$a_2 = R^2 e\,(3d_3 + 6d_4 e + 10d_5 e^2),\qquad a_3 = R^3(d_3 + 4d_4 e + 10d_5 e^2)$$

$$a_4 = R^4(d_4 + 5d_5 e),\qquad a_5 = R^5 d_5$$

5. From (1.2) and (2.1) we have

$$dz = \frac{\lambda}{2\pi i}\, e^{\zeta}\, \frac{du}{u} \tag{5.1}$$

Taking (2.7) and (3.4) into consideration and discarding terms higher than the fifth power, we expand the functions ζ and e^{ζ} into series in powers of u

$$\zeta = au + \mu a^2u^2 + (\mu^2 + d_3)a^3u^3 + (\mu^3 + 3\mu d_3 + d_4)\,a^4u^4 + \\ + (\mu^4 + 6\mu^2 d_3 + 4\mu d_4 + d_5)\,a^5u^5 \tag{5.2}$$

$$e^{\zeta} = b_1u + b_2u^2 + b_3u^3 + b_4u^4 + b_5u^5 \tag{5.3}$$

Here

$$b_1 = a,\qquad b_2 = \left(\mu + \tfrac{1}{2}\right)a^2,\qquad b_3 = \left(\mu^2 + \mu + \tfrac{1}{6} + d_3\right)a^3$$

$$b_4 = [\mu^3 + \tfrac{3}{2}\mu^2 + \left(3d_3 + \tfrac{1}{2}\right)\mu + d_3 + d_4 + \tfrac{1}{24})\,a^4$$

$$b_5 = [\mu^4 + 2\mu^3 + (6d_3 + 1)\,\mu^2 + \left(4d_4 + 4d_3 + \tfrac{1}{6}\right)\mu + \tfrac{1}{2}d_3 + d_4 + d_5]\,a^5$$

Substituting (5.3) into (5.1), integrating and discarding the arbitrary constant, we obtain

$$z = \frac{\lambda}{2\pi i}\left[\ln u + b_1 u + \frac{1}{2} b_2 u^2 + \frac{1}{3} b_3 u^3 + \frac{1}{4} b_4 u^4 + \frac{1}{5} b_5 u^5\right] \quad (5.4)$$

Setting $u = e^{i\theta}$, the equation of the wave profile can be found.

We determine the complex velocity potential in a system of coordinates which is stationary with respect to the fluid at infinite depth from the formula

$$w_1 = w - cz \quad (5.5)$$

Substituting expressions for w and z from (1.2) and (5.4) into (5.5) and omitting the subscript, we obtain

$$w = -\frac{\lambda c}{2\pi i}\left[b_1 u + \frac{1}{2} b_2 u^2 + \frac{1}{3} b_3 u^3 + \frac{1}{4} b_4 u^4 + \frac{1}{5} b_5 u^5\right] \quad (5.6)$$

6. Setting $w = \varphi + i\psi$ in (5.6) and the fluid density $\rho = 1$, we calculate the momentum in one wave period from the formula

$$K = i \oint_L z\, d\varphi \quad (6.1)$$

Integrating (6.1) by parts and noting that $\varphi = 0$ on the segments CD, DE and AE, we obtain

$$K = -i \oint_{L_0} \varphi\, dz, \quad \text{or} \quad K = -\frac{1}{2} i \oint_{L_0} (w + \overline{w})\, dz \quad (6.2)$$

where we carry out the integration along the contour L_0 of a circle of unit radius in the u-plane. Using Formulas (5.4) and (5.6), we find from (6.2)

$$K_y = 0, \quad K_x = -\frac{\lambda^2 c}{2\pi}\left(b_1^2 + \frac{1}{2} b_2^2 + \frac{1}{3} b_3^2 + \frac{1}{4} b_4^2 + \frac{1}{5} b_5^2\right) \quad (6.3)$$

We calculate the kinetic energy of one wave period from the formula

$$T = \frac{1}{2} \oint_L \varphi\, d\psi \quad (6.4)$$

Integrating (6.4) by parts, we find another formula

$$T = -\frac{1}{2} \oint_L \psi\, d\varphi \quad (6.5)$$

Adding (6.4) and (6.5) and dividing by two, we obtain the formula

$$T = \mathrm{Im}\,\frac{1}{4}\oint\limits_{L} \bar{w}\, dw \tag{6.6}$$

Noting that $\varphi = 0$ on the segments CD, DE and AE, we obtain finally

$$T = \mathrm{Im}\,\frac{1}{4}\oint\limits_{L_0} \bar{w} dw \tag{6.7}$$

where we carry out the integration along the contour of a circle of unit radius in the u-plane. Using (5.6), we obtain:

$$T = -\frac{Kc}{2} \tag{6.8}$$

We determine the fluid volume Q displaced per wave period and the static level of the fluid y_0

$$Q = -\frac{K}{c}, \qquad y_0 = -\frac{Q}{\lambda} \tag{6.9}$$

We compute the potential energy per wave period from the formula

$$V = \frac{1}{2} g \int\limits_0^{\lambda} (y - y_0)^2\, dx \tag{6.10}$$

Using Formula (5.4), we obtain

$$V = \frac{\lambda^3 g}{16\pi^2}\Big[b_1^2\left(1 + \frac{3}{2} b_2\right) + \frac{1}{4} b_2^2 (1 + b_4) + \frac{1}{9} b_3^2 + \frac{1}{16} b_4^2 + \frac{1}{25} b_5^2 + + b_1\left(b_2 b_3 + \frac{2}{3} b_3 b_4 + \frac{1}{2} b_4 b_5\right) + \frac{1}{6} b_2 b_3 b_5\Big] - \frac{1}{2}\frac{Q^2}{\lambda} \tag{6.11}$$

In view of the great complexity of the expression for the coefficients of the series (3.4) its convergence could not be proved. The numerical solution indicates that for $a = 0.3$, which corresponds to the ratio $H/\lambda = 0.116$, the series (3.4) and (5.4) converge nicely. In Fig. 1 and Fig. 2 the calculated wave profile and the domain of the function ζ are constructed.

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